THE JOHN–NIRENBERG CONSTANT OF BMO^p, $p > 2$

LEONID SLAVIN AND VASILY VASYUNIN

Abstract. This paper is a continuation of earlier work by the first author who determined the John–Nirenberg constant of $\mathrm{BMO}^p\big((0,1)\big)$ for the range $1 \le p \le 2$. Here, we compute that constant for $p > 2$. As before, the main results rely on Bellman functions for the L^p norms of logarithms of A_∞ weights, but for $p > 2$ these functions turn out to have a significantly more complicated structure than for $1 \le p \le 2$.

1. Preliminaries and main results

For a finite interval J and a function $\varphi \in L^1(J)$, let $\langle\varphi\rangle_J$ denote the average of φ over J with respect to the Lebesgue measure, $\langle\varphi\rangle_J = \frac{1}{|J|}\int_J \varphi$. Take an interval Q and $p > 0$, and let $\mathrm{BMO}(Q)$ be the (factor-)space

$$\mathrm{BMO}(Q) = \{\varphi \in L^1(Q)\colon \quad \|\varphi\|_{\mathrm{BMO}^p(Q)} := \sup_{\text{interval } J\subset Q} \langle|\varphi - \langle\varphi\rangle_J|^p\rangle_J^{1/p} < \infty\}. \tag{1.1}$$

It is a classical fact that all p-based (quasi-)norms are equivalent, which justifies omitting the index p in the left-hand side.

A weight is an almost everywhere positive function. We say that a weight w belongs to $A_\infty(Q)$, $w \in A_\infty(Q)$, if both w and $\log w$ are integrable on Q and the following condition holds:

$$[w]_{A_\infty(Q)} := \sup_{\text{interval } J\subset Q} \langle w\rangle_J e^{-\langle \log w\rangle_J} < \infty.$$

The quantity $[w]_{A_\infty(Q)}$ is called the $A_\infty(Q)$-characteristic of w. When Q is fixed or unimportant, we write simply BMO for $\mathrm{BMO}(Q)$ and A_∞ for $A_\infty(Q)$.

BMO functions are locally exponential integrable. We can state this property in the form of the so-called integral John–Nirenberg inequality, which is a variant of the classical weak-type inequality from [5].

Theorem (John–Nirenberg)**.** *Take $p > 0$. There exists a number $\varepsilon_0(p) > 0$ such that if $\varepsilon \in [0, \varepsilon_0(p))$, Q is an interval, and $\varphi \in \mathrm{BMO}(Q)$ with $\|\varphi\|_{\mathrm{BMO}^p(Q)} \le \varepsilon$, then there is a number $C(\varepsilon, p) > 0$ such that for any interval $J \subset Q$,*

$$\langle e^\varphi\rangle_J \le C(\varepsilon, p) e^{\langle\varphi\rangle_J}. \tag{1.2}$$

We will always use $\varepsilon_0(p)$ to denote the best – largest possible – constant in this theorem and call it the John–Nirenberg constant of BMO^p (on an interval). Likewise, $C(\varepsilon, p)$ will denote the smallest possible constant in (1.2).

Observe that (1.2) means that if $\varphi \in \mathrm{BMO}$, then $e^{\varepsilon\varphi} \in A_\infty$ for all sufficiently small $\varepsilon > 0$. For $\varphi \in \mathrm{BMO}$, let

$$\varepsilon_\varphi = \sup\{\varepsilon\colon e^{\varepsilon\varphi} \in A_\infty\}. \tag{1.3}$$

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In fact, it can be shown that

$$\varepsilon_0(p)=\inf\{\varepsilon_\varphi\colon \|\varphi\|_{\mathrm{BMO}^p}=1\}=\sup\{\varepsilon\colon \forall\varphi,\ \|\varphi\|_{\mathrm{BMO}^p}=1 \implies e^{\varepsilon\varphi}\in A_\infty\}.$$

In this paper, our goal is to compute $\varepsilon_0(p)$ for the case $p>2$. Here are some previous results in that direction: Korenovskii [6] and Lerner [7] computed the analogs for the weak-type John–Nirenberg inequality of $\varepsilon_0(1)$ and $C(\varepsilon,1)$, respectively; in [9], we determined $\varepsilon_0(2)$ and $C(\varepsilon,2)$; in [12], the second author and A. Volberg found all constants in the weak-type inequality for $p=2$; and, finally, in [8], the first author determined $\varepsilon_0(p)$ for $p\in[1,2]$ (including new proofs for the cases $p=1$ and $p=2$) and $C(\varepsilon,p)$ for $p\in(1,2]$ and large enough ε. This last paper built the framework that we follow here, and we refer the reader to it for an in-depth discussion of the tools involved and the differences between the cases $p=2$ and $p\neq 2$.

Let us state the relevant theorem from [8].

Theorem 1.1 ([8])**.** *For* $p\in[1,2]$,

$$\varepsilon_0(p)=\left[\frac{p}{e}\Big(\Gamma(p)-\int_0^1 t^{p-1}e^t\,dt\Big)+1\right]^{1/p}.$$

Furthermore, if $1<p\le 2$, *then for all* $\varepsilon\in[(2-p)\varepsilon_0(p),\varepsilon_0(p))$,

$$C(\varepsilon,p)=\frac{e^{-\varepsilon/\varepsilon_0(p)}}{1-\varepsilon/\varepsilon_0(p)};\tag{1.4}$$

and for all $0\le\varepsilon<\frac{2}{e}$,

$$\frac{e^{-\frac{e}{2}\varepsilon}}{1-\frac{e}{2}\varepsilon}\le C(\varepsilon,1)\le\frac{1}{1-\frac{e}{2}\varepsilon}.\tag{1.5}$$

We can finally complete the picture for all $p\ge 1$. Remarkably, the formula for $\varepsilon_0(p)$ for the case $p>2$ is the same as for $1\le p\le 2$, though it takes much more work to show.

Theorem 1.2. *For* $p>2$,

$$\varepsilon_0(p)=\left[\frac{p}{e}\Big(\Gamma(p)-\int_0^1 t^{p-1}e^t\,dt\Big)+1\right]^{1/p}.\tag{1.6}$$

In contrast with the case $1<p\le 2$, for $p>2$ we do not know the exact $C(\varepsilon,p)$ for any ε. While we could estimate this constant in a manner somewhat similar to (1.5), the estimates we currently have seem much too implicit to be useful, so we omit them.

Without going into details, we mention an important difference between the cases $p\le 2$ and $p>2$. It was shown in [8] that the constant $\varepsilon_0(p)$ is attained in the weak-type John–Nirenberg inequality for $1<p\le 2$ (the case $p=1$ was treated in [6] and [7], while the case $p=2$ had been previously addressed in [12]). However, the method used to show this fact for $p\le 2$ fails for $p>2$, and we do not actually know if the constant is attained (though we conjecture that it is).

On the other hand, another interesting result from [8] does go through for $p>2$. Specifically, we have the following theorem, which extends to $p>2$ the main result of Corollary 1.5 from [8]. It is a sharp lower estimate for the distance in BMO to L^∞ in the spirit of Garnett and Jones [1].

Theorem 1.3. *If* $p>2$, Q *is an interval, and* $\varphi\in\mathrm{BMO}(Q)$, *then*

$$\inf_{f\in L^\infty(Q)}\|\varphi-f\|_{\mathrm{BMO}^p(Q)}\ge\frac{\varepsilon_0(p)}{\min\{\varepsilon_\varphi,\varepsilon_{-\varphi}\}},\tag{1.7}$$

and this inequality is sharp.

As explained in [8], the main idea behind computing $\varepsilon_0(p)$ for $p \neq 2$ is to consider the dual problem: instead of estimating the values of $\|\varphi\|_{\mathrm{BMO}^p}$ for which the exponential oscillation $\langle e^{\varphi-\langle\varphi\rangle}\rangle$ might become unbounded, one estimates from below BMO^p oscillations of logarithms of A_∞ weights and computes their asymptotics as the A_∞-characteristic goes to infinity. This idea is formalized in the following general theorem.

Fix $p>0$. For $C\ge 1$, let

$$\Omega_C=\{x\in\mathbb{R}^2: e^{x_1}\le x_2\le C\,e^{x_1}\}. \tag{1.8}$$

For an interval Q and every $x=(x_1,x_2)\in\Omega_C$, let

$$E_{x,C,Q}=\{\varphi\in L^1(Q)\colon \langle\varphi\rangle_Q=x_1,\ \langle e^\varphi\rangle_Q=x_2,\ [e^\varphi]_{A_\infty(Q)}\le C\}. \tag{1.9}$$

We will call elements of $E_{x,C,Q}$ test functions. Define the following lower Bellman function:

$$\boldsymbol{b}_{p,C}(x)=\inf\{\langle|\varphi|^p\rangle_Q\colon \varphi\in E_{x,C,Q}\}. \tag{1.10}$$

Theorem 1.4 ([8])**.** *Take $p>0$. Assume that there exists a family of functions $\{b_C\}_{C\ge1}$ such that for each C, b_C is defined on Ω_C, $b_C\le\boldsymbol{b}_{p,C}$, and $b_C(0,\cdot)$ is continuous on the interval $[1,C]$. Then*

$$\varepsilon_0^p(p)\ge\limsup_{C\to\infty} b_C(0,C). \tag{1.11}$$

Thus, to estimate $\varepsilon_0(p)$, we need a suitable family $\{b_C\}_C$ of minorants of $\boldsymbol{b}_{p,C}$. Just as was done in [8], we actually find the functions $\boldsymbol{b}_{p,C}$ themselves, for all $p>2$ and all sufficiently large C. We proceed as follows: in Section 2, we construct the so-called Bellman candidate, denoted $b_{p,C}$. This construction is more delicate and more technical than the one in [8], and we briefly discuss the challenges involved. The proof that $b_{p,C}\le\boldsymbol{b}_{p,C}$ constitutes Section 3. It is then an easy matter to prove Theorems 1.2 and 1.3, and it is taken up in Section 4. Finally, in Section 5, we obtain the converse inequality by demonstrating explicit test functions that realize the infimum in (1.10).

2. The construction of the Bellman candidate

For $R>0$, let

$$\Gamma_R=\{x\in\mathbb{R}^2\colon x_2=R\,e^{x_1}\}.$$

Then the domain Ω_C from (1.8) is the region in the plane lying between Γ_1 and Γ_C.

2.1. Discussion and preliminaries.

As mentioned earlier, the construction of the Bellman candidate given here for the case $p>2$ is more involved than those presented in [8] for the cases $p=1$ and $p\in(1,2]$. However, the main goal is the same as before and simple to state: we are building the largest locally convex function b on Ω_C satisfying the boundary condition $b(x_1,e^{x_1})=|x_1|^p$.

Let us briefly explain the similarities and differences between the cases $p\in(1,2]$ and $p>2$ (the case $p=1$ is different from both). In all cases the graph of the candidate b is a convex ruled surface, which means that through each point on the graph there passes a straight-line segment contained in the graph. The domain Ω_C then splits into a collection of subdomains with disjoint interiors, $\Omega_C=\cup_j R_j$, such that b is twice differentiable and satisfies the homogeneous Monge–Ampère equation $b_{x_1x_1}b_{x_2x_2}=b^2_{x_1x_2}$ in the interior of each R_j. In addition, for each subdomain R_j, either b is affine in the whole R_j, or R_j is foliated by straight-line segments connecting two points of the boundary $\Gamma_1\cup\Gamma_C$, and each point $x\in$

$\operatorname{int}(R_j)$ lies on only one such segment. We call such segments Monge–Ampère characteristics of b. Typically, if one knows the characteristics everywhere in Ω_C, one knows the function b.

Thus, to construct the candidate one has to understand how to split Ω_C into subdomains and how to foliate each of them, so that the resulting function b is locally convex. If this is done while ensuring certain compatibility conditions, then b will almost automatically be the *largest* locally convex function with the given boundary conditions, as desired. However, this is, in general, a difficult task, and the situation is further complicated by the fact that the splitting is usually different for different C.

Fortunately, there is now a fairly general theory for building such foliations on special non-convex domains such as ours. It was started in [10] in the context of BMO^2; much developed and systematized in [2] and [3] (still for the parabolic strip of BMO^2); and is now being adapted to general domains, such as Ω_C, in [4]. We also mention the recent paper [11], which formalized the theoretical link between Bellman functions and smallest locally concave (or largest locally convex, as is the case here) functions on the corresponding domains.

A key building block for many Monge–Ampère foliations is the tangential foliation. Let us explain this notion in our setting.

For $C \ge 1$, let $\xi = \xi(C)$ be the unique non-negative solution of the equation

$$e^{-\xi} = C(1-\xi): \quad 0 \le \xi < 1.$$

Note that $\xi(1) = 0$ and that ξ is strictly increasing with $\lim_{C\to\infty} \xi(C) = 1$. Let

$$k(z) = \frac{e^z}{1-\xi}, \quad z \in \mathbb{R}, \tag{2.1}$$

and define a new function $u = u(x)$ on Ω_C by the implicit formula

$$x_2 = k(u)(x_1 - u) + e^u. \tag{2.2}$$

This function has simple geometrical meaning, illustrated in Figure 1: if one draws the one-sided tangent to Γ_C that passes through x, so that the point of tangency is to the right of x, then this tangent intersects Γ_1 at the point (u, e^u), while the point of tangency is $(u+\xi, Ce^{u+\xi})$. In particular $u(0, C) = -\xi$. (We note that in [8], ξ and u were called ξ^+ and u^+, respectively).

In the case $1 < p \le 2$, if C was large enough, all of Ω_C was foliated by the tangents (2.2), for $u \in (-\infty, \infty)$; thus, we did not have to split it into subdomains. However, for $p > 2$, this uniform tangential foliation fails to yield a locally convex function on the whole Ω_C, for any C. What actually happens — and, again, only for sufficiently large C — is shown on Figure 2 later in this section. There we have two tangentially foliated subdomains, R_1 and R_3, which are linked by a special "transition regime" consisting of two more subdomains: R_2, where the candidate is affine and the foliation is thus degenerate, and R_4, where the characteristics are chords connecting two points of Γ_1. (In recent Bellman-function literature, these two particular shapes are called "trolleybus" and "cup", respectively; see [2, 3, 4].) This transition regime shrinks as C grows, but never disappears. To show how all this fits together, we need some technical preparation.

2.2. **Technical lemmas.**

Lemma 2.1.

(1) *If* $w > 0$ *and* $v \in \big(-w, -w\,\frac{p-1}{p}\big)$, *then*

$$\frac{w^{p-1} + (-v)^{p-1}}{e^w - e^v} < (p-1)(-v)^{p-2}e^{-v}. \tag{2.3}$$

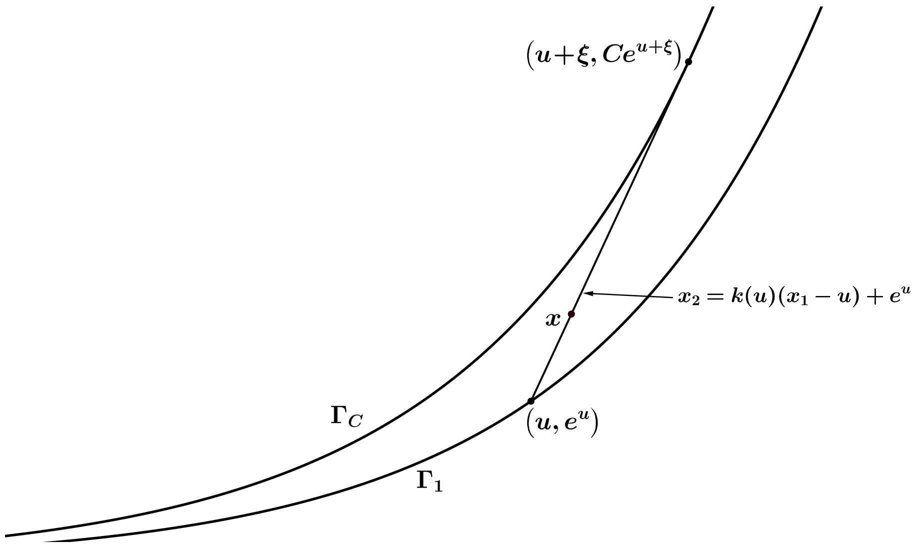

FIGURE 1. The geometric meaning of $u(x)$ and ξ

(2) *If* $0 < w \le \frac{p-2}{p-1}$ *and* $v \in (-w, 0)$, *then*

$$\frac{w^{p-1} + (-v)^{p-1}}{e^w - e^v} < (p-1)w^{p-2}e^{-w}. \tag{2.4}$$

Proof. For Part (1), note that $e^{w-v} - 1 \ge w - v > 0$, and so it is sufficient to check that

$$w^{p-1} + (-v)^{p-1} < (p-1)(-v)^{p-2}(w-v)\,.$$

Put $\theta = -\frac{v}{w}$; then this inequality becomes

$$(p-2)\theta^{p-1} + (p-1)\theta^{p-2} - 1 > 0\,, \qquad \frac{p-1}{p} < \theta < 1\,.$$

The left-hand side is increasing in θ, and equals $2\frac{(p-1)^p}{p^{p-1}} - 1$ when $\theta = \frac{p-1}{p}$. In turn, this is an increasing function of p, equal to 0 at $p = 2$.

For Part (2), observe that since $w > -v$, and $1 - w \ge 1 - \frac{p-2}{p-1} = \frac{1}{p-1}$, we have

$$1 - e^{-(w-v)} > (w-v)\Big[1 - \frac{1}{2}(w-v)\Big] > (w-v)(1-w) \ge \frac{w-v}{p-1},$$

and (2.4) follows from the obvious relation $w^{p-1} + (-v)^{p-1} < w^{p-2}(w-v)$. □

For any $v < 0$ and $w > 0$, let

$$r(v,w) = \frac{e^w - e^v}{w - v}, \qquad q(v,w) = \frac{w^p - (-v)^p}{w - v}. \tag{2.5}$$

Lemma 2.2. *For each* $w \in \big(0, \frac{p-2}{3p}\big)$, *there exists a unique* $v \in \big(-w, -w\,\frac{p-1}{p}\big)$ *such that*

$$\frac{q(v,w) + p(-v)^{p-1}}{r(v,w) - e^v} = \frac{pw^{p-1} - q(v,w)}{e^w - r(v,w)} = p\,\frac{w^{p-1} + (-v)^{p-1}}{e^w - e^v}. \tag{2.6}$$

Proof. Observe that it is enough to show only the first equality in (2.6), as the second one then follows by elementary rearrangement. In turn, this first equality is equivalent to the statement

$$F(v,w) := (e^w - e^v)(w^p - (-v)^p - pw^{p-1} - p(-v)^{p-1}) + p(w-v)(w^{p-1}e^v + (-v)^{p-1}e^w) = 0. \tag{2.7}$$

Assume $w \in (0, \frac{p-2}{3p})$ and let $\lambda = \frac{p-1}{p}$. To show that there exists $v \in (-w, -\lambda w)$ such that $F(v,w) = 0$, we compare the signs of $F(-w,w)$ and $F(-\lambda w, w)$.

Since $F(-w,w) = 4pw^{p-1}(w\cosh w - \sinh w) > 0$, we want to check that $F(-\lambda w, w) < 0$. To that end,

$$F(-\lambda w, w) = (e^w - e^{-\lambda w})\big[(1-\lambda^p)w^p - p(1+\lambda^{p-1})w^{p-1}\big] + p(1+\lambda)w^p\big[e^{-\lambda w} + \lambda^{p-1}e^w\big],$$

and the inequality $F(-\lambda w, w) < 0$ is equivalent to

$$\frac{(1+\lambda)w}{e^{(1+\lambda)w} - 1} - 1 + \frac{1 + p\lambda^{p-1} + (p-1)\lambda^p}{p(1+\lambda^{p-1})} w < 0.$$

Let

$$\psi(t) = \frac{t}{e^t - 1} - 1 + \frac{1 + p\lambda^{p-1} + (p-1)\lambda^p}{p(1+\lambda)(1+\lambda^{p-1})} t\,.$$

We would like to show that $\psi(t) < 0$ for $t \in (0, \frac{p-2}{3p}(1+\lambda))$. Note that for $t > 0$,

$$e^t > 1 + t + \frac{1}{2}t^2 \qquad \Longrightarrow \qquad \frac{t}{e^t - 1} - 1 < -\frac{t}{t+2}\,.$$

Therefore, for $t > 0$,

$$\psi(t) < -t\Big[\frac{1}{t+2} - \frac{1 + p\lambda^{p-1} + (p-1)\lambda^p}{p(1+\lambda)(1+\lambda^{p-1})}\Big],$$

and it is sufficient to check that

$$\frac{p-2}{3p}(1+\lambda) < \frac{p(1+\lambda)(1+\lambda^{p-1})}{1 + p\lambda^{p-1} + (p-1)\lambda^p} - 2 = \frac{(p-2)(1-\lambda^p) + p\lambda(1-\lambda^{p-2})}{1 + p\lambda^{p-1} + (p-1)\lambda^p}.$$

Since $(p-2)(1-\lambda^p) + p\lambda(1-\lambda^{p-2}) > (p-2)(1+\lambda)(1-\lambda^{p-1})$ and $\lambda^p < \lambda^{p-1}$, it is enough to verify that

$$3 > \frac{1/p + (2-1/p)\lambda^{p-1}}{1-\lambda^{p-1}} = \frac{2}{1-\lambda^{p-1}} - 2 + 1/p.$$

This is so because the right-hand side is decreasing in p and equals $\frac{5}{2}$ when $p = 2$. This proves that the desired v exists for each w.

To show that v is unique, we differentiate the function F with respect to v. This derivative can be written as follows:

$$\begin{aligned} F_v(v,w) &= \big(e^w(w-v) - e^w + e^v\big)e^v\left(\frac{pw^{p-1}(w-v) - w^p + (-v)^p}{e^w(w-v) - e^w + e^v} - p(p-1)(-v)^{p-2}e^{-v}\right) \\ &= \big(e^w(w-v) - e^w + e^v\big)e^v\left(p\,\frac{w^{p-1} + (-v)^{p-1}}{e^w - e^v} - p(p-1)(-v)^{p-2}e^{-v}\right), \end{aligned}$$

where we used the second equality in (2.6). Now, the first factor is positive, because the function $t \mapsto e^t$ is strictly convex, while the last factor is negative by (2.3). Therefore, $F_v(v,w)$ is negative for any root v of the equation $F(v,w) = 0$ that lies in the interval $\big(-w, -w\,\frac{p-1}{p}\big)$, which is possible only when such a root is unique. $\square$

From now on, in using v and w we will always presume that $w \in (0, \frac{p-2}{3p})$, $v \in (-w, -w\,\frac{p-1}{p})$, and the pair $\{v, w\}$ is related by equation (2.6). For such v and w, each of the three equal quantities in (2.6) is a function of w, and it is convenient to give them a common name. Let

$$D(w) = \frac{q(v,w) + p(-v)^{p-1}}{r(v,w) - e^v} = \frac{pw^{p-1} - q(v,w)}{e^w - r(v,w)} = p\,\frac{w^{p-1} + (-v)^{p-1}}{e^w - e^v}, \tag{2.8}$$

and D is a function of w defined on the interval $\big(0, \frac{p-2}{3p}\big)$. Let us list some of its properties.

Lemma 2.3. *We have*

$$D(w) < p(p-1)(-v)^{p-2}e^{-v} \tag{2.9}$$

and

$$D(w) < p(p-1)w^{p-2}e^{-w}. \tag{2.10}$$

Furthermore, $D' > 0$ *on* $\left(0, \frac{p-2}{3p}\right)$.

Proof. Inequalities (2.9) and (2.10) come directly from (2.3) and (2.4), respectively (note that $\frac{p-2}{3p} < \frac{p-2}{p-1}$, so (2.4) applies).

To check the sign of D', we will treat q and r as functions of w and use the prime to indicate the total derivative with respect to w. Thus, $q' = q_w + q_v v_w$ and $r' = r_w + r_v v_w$, where v_w can be computed from equation (2.7). Also denote $f(w) := w^p$ and $g(w) := e^w$.

We will need the simple but key fact that equation (2.6) can be written as

$$\frac{q'}{r'} = \frac{q - f'}{r - g'} = D. \tag{2.11}$$

Using this identity, we have

$$\begin{aligned} D' = \Big(\frac{q-f'}{r-g'}\Big)' &= \frac{(q'-f'')(r-g') - (q-f')(r'-g'')}{(r-g')^2} \\ &= \frac{g''(q-f') - f''(r-g')}{(r-g')^2} = \frac{g''}{r-g'}\Big(\frac{q-f'}{r-g'} - \frac{f''}{g''}\Big). \end{aligned}$$

Since g is strictly convex, we have $g'' > 0$ and $r - g' < 0$. On the other hand, the expression in parentheses is negative by (2.10). ☐

For $p > 2$, let

$$\xi_0(p) = 1 - \frac{1}{3^{p+2}\Gamma(p)}, \qquad C_0(p) = \frac{e^{-\xi_0(p)}}{1 - \xi_0(p)}. \tag{2.12}$$

Lemma 2.4. *Assume* $\xi > \xi_0(p)$. *Let*

$$c_1 = \xi\big[e(1-\xi)\Gamma(p-1)\big]^{1/(p-2)}, \qquad c_2 = \xi\big[2e(1-\xi)\Gamma(p)\big]^{1/(p-2)}.$$

Then the equation

$$\Big(\frac{1}{\xi} - 1\Big)\int_w^\infty s^{p-2}e^{-s/\xi}\,ds - w^{p-2}e^{-w/\xi} = 0 \tag{2.13}$$

has a unique solution w_* *in the interval* $(0, c_1)$.

Furthermore, the equation

$$\Big(\frac{1}{\xi} - 1\Big)p(p-1)e^{w(1/\xi-1)}\int_w^\infty s^{p-2}e^{-s/\xi}\,ds = D(w) \tag{2.14}$$

has a unique solution $\overline{w}$ *in the interval* (w_*, c_2).

Proof. First observe that $c_1 < c_2 < \xi$. The first inequality is trivial, while the second is equivalent to $\xi > 1 - \frac{1}{2e\Gamma(p)}$, which is clearly satisfied by the assumption $\xi > \xi_0(p)$. Second, we have $c_2 < \frac{p-2}{3p}$. Indeed, this inequality is equivalent to

$$\xi > 1 - \frac{\big(1 - \frac{2}{p}\big)^{p-2}}{2e\,(3\xi)^{p-2}\Gamma(p)}.$$

Since $\xi<1$ and $\big(1-\frac{2}{p}\big)^{p-2}>e^{-2}$, this inequality is weaker than $\xi>1-\frac{9}{2e^3 3^p\Gamma(p)}$, which is in turn weaker than $\xi>\xi_0(p)$.

Consider equation (2.13). When $w=0$, the left-hand side of (2.13) is positive. For $w=c_1$,

$$\Big(\frac{1}{\xi}-1\Big)\int_{c_1}^\infty s^{p-2}e^{-s/\xi}\,ds-c_1^{p-2}e^{-c_1/\xi}=(1-\xi)\xi^{p-2}\int_{c_1/\xi}^\infty s^{p-2}e^{-s}\,ds-c_1^{p-2}e^{-c_1/\xi}$$
$$<(1-\xi)\xi^{p-2}\Gamma(p-1)-c_1^{p-2}e^{-c_1/\xi}=\xi^{p-2}(1-\xi)\Gamma(p-1)\big(1-e^{1-c_1/\xi}\big)<0,$$

since $c_1<\xi$. Thus, a solution $w_*\in(0,c_1)$ exists. To prove that it is unique, we note that the left-hand side of (2.13) is decreasing in w for $w\in(0,p-2)$, and that $c_1<p-2$.

Turning now to (2.14), for $w=w_*$ we have, by (2.13) and (2.10),

$$\Big(\frac{1}{\xi}-1\Big)p(p-1)e^{w_*(1/\xi-1)}\int_{w_*}^\infty s^{p-2}e^{-s/\xi}\,ds=p(p-1)w_*^{p-2}e^{-w_*}>D(w_*).$$

Observe that for any w, since $1-e^{v-w}<1-e^{-2w}<2w$,

$$D(w)=p\,\frac{w^{p-1}+(-v)^{p-1}}{e^w-e^v}>pe^{-w}\frac{w^{p-1}}{1-e^{v-w}}>\frac{1}{2}\,pe^{-w}w^{p-2}.$$

Therefore, putting $w=c_2$ in the left-hand side of (2.14) we get

$$\Big(\frac{1}{\xi}-1\Big)p(p-1)e^{c_2(1/\xi-1)}\int_{c_2}^\infty s^{p-2}e^{-s/\xi}\,ds<(1-\xi)\xi^{p-2}p\Gamma(p)e^{1-c_2}=\frac{1}{2}\,pe^{-c_2}c_2^{p-2}<D(c_2),$$

and, hence, a solution $\overline{w}\in(w_*,c_2)$ exists. To prove uniqueness, observe that the derivative of the left-hand side of (2.14) is a positive multiple of the left-hand side of (2.13); thus, it equals zero at $w=w_*$ and is decreasing for $w\in(0,p-2)$; in particular, it is negative for $w>w_*$. Therefore, the left-hand side of (2.14) is decreasing in w for $w\ge w_*$, while by Lemma 2.3, the right-hand side is increasing. $\square$

Remark 2.5. In what follows, in addition to $\overline{w}$, we will also use $\bar v$, which is the unique solution of the equation $F(v,\overline{w})=0$ guaranteed by Lemma 2.2.

2.3. **The Bellman candidate.** As mentioned earlier, we now split domain Ω_C into four subdomains, $\Omega_C=\cup_{j=1}^4R_j$. In addition to the numbers $\bar v$ and $\overline{w}$ given by Lemma 2.14 and Remark 2.5, the definition below uses the function k from (2.1) and function r from (2.5). The splitting is pictured in Figure 2.

(2.15)
$$R_1=\big\{x\in\Omega_C\colon x_2\le k(\overline{w})(x_1-\overline{w})+e^{\overline{w}}\big\}\cup\big\{x\in\Omega_C\colon x_1\ge\overline{w}+\xi\big\};$$
$$R_2=\big\{x\in\Omega_C\colon x_2\le k(\bar v)(x_1-\bar v)+e^{\bar v},\ x_2\ge r(\bar v,\overline{w})(x_1-\overline{w})+e^{\overline{w}},\ x_2\ge k(\overline{w})(x_1-\overline{w})+e^{\overline{w}}\big\}$$
$$\cup\big\{x\in\Omega_C\colon \bar v+\xi\le x_1\le\overline{w}+\xi,\ x_2\ge k(\bar v)(x_1-\bar v)+e^{\bar v},\ x_2\ge k(\overline{w})(x_1-\overline{w})+e^{\overline{w}}\big\};$$
$$R_3=\big\{x\in\Omega_C\colon x_1\le\bar v+\xi,\ x_2\ge k(\bar v)(x_1-\bar v)+e^{\bar v}\big\};$$
$$R_4=\big\{x\in\Omega_C\colon x_2\le r(\bar v,\overline{w})(x_1-\overline{w})+e^{\overline{w}}\big\}.$$

Our Bellman candidate will have a different expression in each of the four subdomains, requiring several auxiliary objects. For $z\in\mathbb{R}$, let

$$m_1(z)=\frac{p}{\xi}\,e^{z/\xi}\int_z^\infty s|s|^{p-2}e^{-s/\xi}\,ds, \tag{2.16}$$

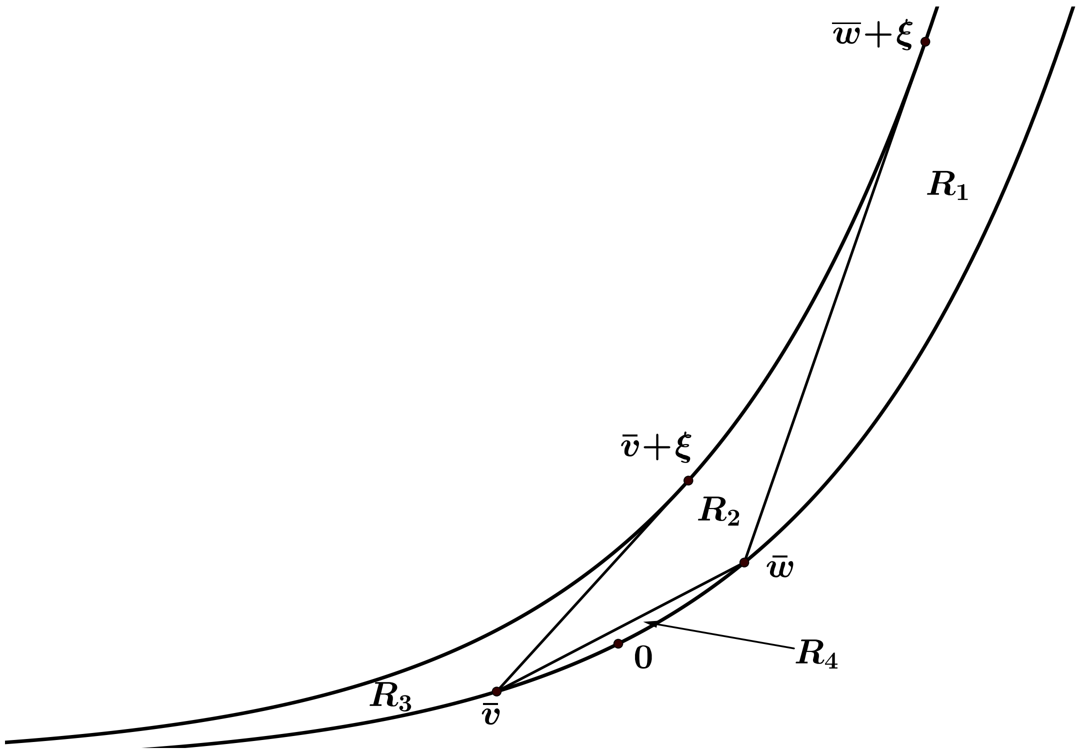

FIGURE 2. The splitting of Ω_C for sufficiently large C

and for $z < \bar{v}$, let

$$m_3(z) = -\frac{p}{\xi}\, e^{z/\xi} \int_z^{\bar{v}} (-s)^{p-1} e^{-s/\xi}\, ds + e^{(z-\bar{v})/\xi} \left(e^{\bar{v}-\overline{w}} \big(m_1(\overline{w}) - p\overline{w}^{p-1} \big) - p(-\bar{v})^{p-1} \right). \tag{2.17}$$

The following intuitive lemma, whose simple proof is left to the reader, defines two new functions on R_4.

Lemma 2.6. *For each $x = (x_1, x_2) \in R_4$ there exists a unique pair $\{v, w\}$ satisfying* (2.7) *and such that the line segment connecting the points (w, e^w) and (v, e^v) passes through x. Thus,*

$$x_2 = r(v, w)(x_1 - w) + e^w.$$

In the special case $x = (0, 1)$ this segment degenerates into a point: $v = w = 0$.

From here on, we reserve the symbols w and v for the two functions on R_4 given by this lemma: $w = w(x)$ and $v = v(x)$; see Figure 3.

Finally, here is our complete Bellman candidate. For $p > 2$ and $C > C_0(p)$, let

$$b_{p,C}(x) = \begin{cases} m_1(u)(x_1 - u) + u^p, & x \in R_1, \\ q(\bar{v}, \overline{w})\,(x_1 - \overline{w}) + \overline{w}^p + \frac{m_1(\overline{w}) - q(\bar{v}, \overline{w})}{k(\overline{w}) - r(\bar{v}, \overline{w})} \big(x_2 - r(\bar{v}, \overline{w})(x_1 - \overline{w}) - e^{\overline{w}} \big), & x \in R_2, \\ m_3(u)(x_1 - u) + (-u)^p, & x \in R_3, \\ q\big(v, w\big)\,(x_1 - w) + w^p, & x \in R_4. \end{cases} \tag{2.18}$$

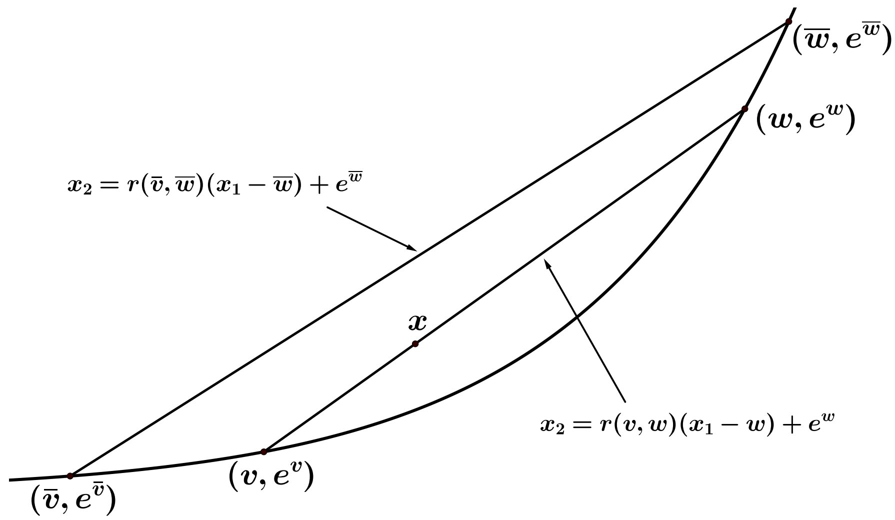

FIGURE 3. The subdomain R_4 and the geometric meaning of $v(x)$ and $w(x)$

Recall that here $u = u(x)$ is given by (2.2); $v = v(x)$ and $w = w(x)$ have just been defined in Lemma 2.6; k is given by (2.1); r and q are given by (2.5); m_1 is given by (2.16); and m_3 is given by (2.17). In addition, $\overline{w}$ was defined in Lemma 2.4 as the solution of equation (2.14), while $\bar{v}$ was defined in the remark following that lemma as the unique solution of equation $F(v, \overline{w}) = 0$ with F given by equation (2.7).

The next section presents the main theorem relating the candidate $b_{p,C}$ and the Bellman function $\boldsymbol{b}_{p,C}$ from (1.10).

3. The main Bellman theorem and a proof of the lower estimate

The following result is the principal ingredient in the proofs of Theorems 1.2 and 1.3.

Theorem 3.1. *If $p > 2$ and $C > C_0(p)$, then*

$$\boldsymbol{b}_{p,C} = b_{p,C} \quad in \quad \Omega_C. \tag{3.1}$$

As is common, we split the proof of Theorem 3.1 in two parts: the so-called direct inequality $\boldsymbol{b}_{p,C} \geq b_{p,C}$ and its converse.

Lemma 3.2. *If $p > 2$ and $C > C_0(p)$, then*

$$\boldsymbol{b}_{p,C} \geq b_{p,C} \quad in \quad \Omega_C. \tag{3.2}$$

Lemma 3.3. *If $p > 2$ and $C > C_0(p)$, then*

$$\boldsymbol{b}_{p,C} \leq b_{p,C} \quad in \quad \Omega_C. \tag{3.3}$$

The proofs of Theorems 1.2 and 1.3 use only Lemma 3.2, which we prove in this section. For the sake of completeness, we will also show that the infimum in the definition of the Bellman function is attained at every point in Ω_C, and our candidate is in fact the Bellman function. This is done in Section 5 where we prove Lemma 3.3.

The analog of Lemma 3.2 for $p \in [1, 2]$ was proved in Section 5 of [8]. In fact, the proof given there did not depend on the specific range of p used. Rather, its main ingredient was showing that $b_{p,C}$ is locally convex in Ω_C, i.e., convex along every line segment contained in Ω_C. More precisely, the main result of Lemma 5.1 from [8] can be restated as follows.

Lemma 3.4 ([8])**.** *Fix $p > 0$ and assume that for some $C(p) \geq 1$ there is a family of functions $\{b_{p,C}\}_{C \geq C(p)}$ satisfying the following conditions for each C :*

(1) *$b_{p,C}$ is locally convex in Ω_C;*
(2) *$b_{p,C}$ is continuous in Ω_C;*
(3) *For each $x \in \Omega_C$,*

$$\lim_{c \searrow C} b_{p,c}(x) = b_{p,C}(x);$$

(4) *For each $s \in \mathbb{R}$, $b_{p,C}(s, e^s) = |s|^p$.*

Then for all $C \geq C(p)$,

$$\boldsymbol{b}_{p,C} \geq b_{p,C} \quad \text{in} \quad \Omega_C.$$

It is routine to check that conditions (2)-(4) are satisfied for $b_{p,C}$ from (2.18). Therefore, Lemma 3.2 will be proved once we have established the following result.

Lemma 3.5. *For $p > 2$ and $C > C_0(p)$, the function $b_{p,C}$ is locally convex in Ω_C.*

Let us fix $p > 2$ and $C > C_0(p)$ and through the end of this section write simply b for $b_{p,C}$. Before proving Lemma 3.5, let us collect several useful facts from earlier work. First, as explained in [10] and [8], showing that b is locally convex in Ω_C is the same as showing that it is locally convex in each subdomain R_k and that b_{x_2} is increasing in x_2 across shared boundaries between subdomains. Second, in R_1 and R_3, b has the form

$$b(x) = m(u)(x_1 - u) + |u|^p,$$

where m stands for m_1 or m_3, respectively, and in each case satisfies the differential equation

$$m'(u) = \frac{1}{\xi}(m(u) - pu|u|^{p-2}), \tag{3.4}$$

and $u = u(x)$ is given by (2.2). As shown in [8], in such a case we have

$$b_{x_2} = m'(u)e^{-u}(1 - \xi) \tag{3.5}$$

and also

$$b_{x_1x_1}b_{x_2x_2} = b_{x_1x_2}^2, \qquad \operatorname{sgn} b_{x_2x_2} = \operatorname{sgn}\big(m'(u) - m''(u)\big). \tag{3.6}$$

Therefore, to show that b is locally convex in R_1 and R_3 we simply need to show that $m_1'(u) - m_1''(u) > 0$ in R_1 and $m_3'(u) - m_3''(u) > 0$ in R_3.

Proof of Lemma 3.5. We first show local convexity of b in each subdomain R_k.

In R_1, a direct computation gives

$$\frac{\xi^2}{p(p-1)}(m_1'(u) - m_1''(u))e^{-u/\xi} = \xi u^{p-2}e^{-u/\xi} - (1 - \xi)\int_u^\infty e^{-s/\xi}s^{p-2}\,ds =: H_1(u),$$

where $u \geq \overline{w}$. We have

$$H_1'(u) = \xi u^{p-3}e^{-u/\xi}(p - 2 - u).$$

Therefore, H_1 is increasing for $u \in (0, p-2)$ and decreasing for $u > p - 2$. Since $H_1(u) \to 0$ as $u \to \infty$, to show that $H_1(u) > 0$ for $u \geq \overline{w}$, it suffices to show that $H_1(\overline{w}) > 0$. This immediately follows by applying first (2.14) and then (2.10) with $w = \overline{w}$:

$$(1 - \xi)\int_{\overline{w}}^\infty e^{-s/\xi}s^{p-2}\,ds = \frac{\xi e^{\overline{w}(1-1/\xi)}}{p(p-1)}\,D(\overline{w}) < \xi\overline{w}^{p-2}e^{-\overline{w}/\xi}.$$

Therefore, $b_{x_2x_2} > 0$ in R_1 and so b is locally convex in this subdomain.

In R_2, b is affine and thus locally convex.

In R_3, we compute

$$\begin{aligned}\frac{\xi^2}{p(p-1)}&(m_3'(u)-m_3''(u))e^{-u/\xi}\\&=\xi(-u)^{p-2}e^{-u/\xi}-(1-\xi)\Big(\int_u^{\bar v}e^{-s/\xi}(-s)^{p-2}\,ds+e^{(\overline{w}-\bar v)(1/\xi-1)}\int_{\overline{w}}^{\infty}e^{-s/\xi}s^{p-2}\,ds\Big)\\&=:H_3(u),\end{aligned}$$

where $u\le\bar v$. We have

$$H_3'(u)=\xi(-u)^{p-3}e^{-u/\xi}(u-p+2)<0,$$

and so to show that $H_3(u)>0$, it is enough to show that $H_3(\bar v)>0$. Similarly to the case of H_1, this follows from an application of (2.14) and then of (2.9) with $v=\bar v$:

$$(1-\xi)e^{(\overline{w}-\bar v)(1/\xi-1)}\int_{\overline{w}}^{\infty}e^{-s/\xi}s^{p-2}\,ds=\frac{\xi e^{-\bar v(1/\xi-1)}}{p(p-1)}D(\overline{w})<\xi(-\bar v)^{p-2}e^{-\bar v/\xi}.$$

Thus, b is locally convex in R_3.

Let us state the result for R_4 separately.

Lemma 3.6. *b is convex in R_4.*

Proof. In R_4, b is given by

$$b(x)=q(v,w)(x_1-w)+f(w),\qquad x_2=r(v,w)(x_1-w)+g(w),$$

where, as in the proof of Lemma 2.3, we write $f(w)=w^p$ and $g(w)=e^w$. Let us also, as we did there, use the prime to indicate the total derivative with respect to w.

To show that b is convex, we show that $b_{x_1x_1}b_{x_2x_2}=b^2_{x_1x_2}$ and $b_{x_2x_2}>0$ in the interior of R_4. Differentiating gives

$$w_{x_1}=\frac{-r}{r'(x_1-w)-r+g'},\qquad w_{x_2}=\frac{1}{r'(x_1-w)-r+g'},$$

and

$$b_{x_1}=\big[q'(x_1-w)-q+f'\big]w_{x_1}+q=-r\,\frac{q'(x_1-w)-q+f'}{r'(x_1-w)-r+g'}+q=-rD+q,$$

where we used (2.11). Similarly,

$$b_{x_2}=\big[q'(x_1-w)-q+f'\big]w_{x_2}=\frac{q'(x_1-w)-q+f'}{r'(x_1-w)-r+g'}=D. \tag{3.7}$$

Therefore,

$$b_{x_1x_1}=-rD'w_{x_1},\quad b_{x_1x_2}=-rD'w_{x_2},\quad b_{x_2x_2}=D'w_{x_2},$$

and, since, $w_{x_1}=-rw_{x_2}$, we see that $b_{x_1x_1}b_{x_2x_2}=b^2_{x_1x_2}$.

Furthermore, since by Lemma 2.3, $D'>0$ and since it is clear from geometry that $w_{x_2}>0$, we have $b_{x_2x_2}>0$, which completes the proof. □

To finish the proof of Lemma 3.5, we need to verify that b_{x_2} is increasing in x_2 across boundaries between subdomains. We can write this requirement symbolically as:

$$b_{x_2}\Big|_{R_1,u=\overline{w}}\le b_{x_2}\Big|_{R_2},\qquad b_{x_2}\Big|_{R_4,w=\overline{w}}\le b_{x_2}\Big|_{R_2},\qquad b_{x_2}\Big|_{R_2}\le b_{x_2}\Big|_{R_3,u=\bar v}.$$

In fact, all three statements hold with equality (which implies that b is of class C^1 in the interior of Ω_C, though we will not use this fact).

From (3.7), we have $b_{x_2}\big|_{R_4, w=\overline{w}} = D(\overline{w})$. Using, in order, (3.5), (3.4), (2.16), integration by parts, and (2.14) gives:

$$b_{x_2}\Big|_{R_1, u=\overline{w}} = m_1'(\overline{w})e^{-\overline{w}}(1-\xi) = \frac{1}{\xi}(1-\xi)p(p-1)e^{\overline{w}(1/\xi-1)}\int_{\overline{w}}^{\infty} s^{p-2}e^{-s/\xi}\,ds = D(\overline{w}). \tag{3.8}$$

A very similar calculation, but using (2.17) in place of (2.16), gives $b_{x_2}\big|_{R_3, u=\bar{v}} = D(\overline{w})$.

Finally,

$$b_{x_2}\Big|_{R_2} = \frac{m_1(\overline{w}) - q(\bar{v}, \overline{w})}{k(\overline{w}) - r(\bar{v}, \overline{w})}.$$

By (3.4) and (3.8),

$$m_1(\overline{w}) = \xi m_1'(\overline{w}) + p\overline{w}^{p-1} = \frac{\xi}{1-\xi}\, e^{\overline{w}}D(\overline{w}) + p\overline{w}^{p-1}.$$

Therefore,

$$b_{x_2}\Big|_{R_2} = \frac{\frac{\xi}{1-\xi}\, e^{\overline{w}}D(\overline{w}) + p\overline{w}^{p-1} - q(\bar{v}, \overline{w})}{\frac{1}{1-\xi}\, e^{\overline{w}} - r(\bar{v}, \overline{w})} = \frac{\frac{\xi}{1-\xi}\, e^{\overline{w}}D(\overline{w}) + \big(e^{\overline{w}} - r(\bar{v}, \overline{w})\big)D(\overline{w})}{\frac{1}{1-\xi}\, e^{\overline{w}} - r(\bar{v}, \overline{w})} = D(\overline{w}).$$

The proof is complete. □

We are now in a position to prove the main theorems stated in Section 1.

4. Proofs of Theorems 1.2 and 1.3

We will need two auxiliary results from [8].

For $p > 0$, let

$$\omega(p) = \left[\frac{p}{e}\Big(\Gamma(p) - \int_0^1 t^{p-1}e^t\,dt\Big) + 1\right]^{1/p}.$$

Lemma 4.1 ([8])**.** *Let $\varphi_0(t) = \log(1/t)$, $t \in (0,1)$. Then*

$$\varepsilon_{\varphi_0} = 1, \qquad \varepsilon_{-\varphi_0} = \infty. \tag{4.1}$$

If $p \geq 1$, then

$$\|\varphi_0\|_{\mathrm{BMO}^p((0,1))} = \omega(p). \tag{4.2}$$

Consequently,

$$\varepsilon_0(p) \leq \omega(p). \tag{4.3}$$

Lemma 4.2 ([8])**.** *Let φ be a non-constant* BMO *function. For $\varepsilon \in [0, \varepsilon_\varphi)$, let $F(\varepsilon) = [e^{\varepsilon\varphi}]_{A_\infty}$. Then F is a strictly increasing, continuous function on $[0, \varepsilon_\varphi)$, and $\lim_{\varepsilon\to\varepsilon_\varphi} F(\varepsilon) = \infty$.*

Proof of Theorem 1.2. We use Theorem 1.4 with $b_C = b_{p,C}$ given by formula 2.18. By Lemma 3.2, $b_C \leq \boldsymbol{b}_{p,C}$, as required.

We need to compute $b_{p,C}(0, C)$. Note that by Lemma 2.4, $\overline{w} < \xi$, and, thus, $\bar{v} > -\overline{w} > -\xi$ by Lemma 2.2. Therefore, the point $(0, C)$ is in subdomain R_3 and, since $u(0, C) = -\xi$,

$$b_{p,C}(0, C) = m_3(-\xi)\xi + \xi^p.$$

Now, $m_3(-\xi)$ is given by (2.17):

$$m_3(-\xi) = -\frac{p}{\xi}\, e^{-1}\int_{-\xi}^{\bar{v}} (-s)^{p-1}e^{-s/\xi}\,ds + e^{(-\xi-\bar{v})/\xi}\left(e^{\bar{v}-\overline{w}}\big(m_1(\overline{w}) - p\overline{w}^{p-1}\big) - p(-\bar{v})^{p-1}\right).$$

By Lemma 2.4, $\overline{w}\in(0,c_2)$ with $c_2\to0$ as $\xi\to1$. By Lemma 2.2, $\bar v\in(-\overline{w},0)$. Therefore,

$$\lim_{\xi\to1}\bar v=\lim_{\xi\to1}\overline{w}=0$$

and

$$\lim_{C\to\infty}b_{p,C}(0,C)=\lim_{\xi\to1}\big(m_3(-\xi)\xi+\xi^p\big)=-\frac{p}{e}\int_{-1}^{0}(-s)^{p-1}e^{-s}\,ds+e^{-1}m_1(0)+1=\omega^p(p),$$

where we used (2.16), whereby $m_1(0)=p\Gamma(p)$.

Hence, by Theorem 1.4, $\varepsilon_0(p)\ge\omega(p)$, and Lemma 4.1 now finishes the proof. □

The proof of Theorem 1.3 below is a variation of the argument for Corollary 1.5 in [8]; the proof of sharpness, using function φ_0 from Lemma 4.1, is exactly the same and we omit it.

Proof of Theorem 1.3. Take any $\varphi\in\mathrm{BMO}(Q)$. Without loss of generality, assume $\varepsilon_\varphi<\infty$. For $\varepsilon\in[0,\varepsilon_\varphi)$, let $F(\varepsilon)=[e^{\varepsilon\varphi}]_{A_\infty(Q)}$. By Lemma 4.2, for sufficiently large ε we have $F(\varepsilon)\ge C_0(p)$. Therefore, for any subinterval J of Q,

$$\langle|\varepsilon\varphi-\langle\varepsilon\varphi\rangle_J|^p\rangle_J\ge\boldsymbol{b}_{p,F(\varepsilon)}\big(0,\langle e^{\varepsilon\varphi-\langle\varepsilon\varphi\rangle_J}\rangle_J\big)\ge b_{p,F(\varepsilon)}\big(0,\langle e^{\varepsilon\varphi-\langle\varepsilon\varphi\rangle_J}\rangle_J\big).$$

Take a sequence $\{J_n\}$ such that $\langle e^{\varepsilon\varphi-\langle\varepsilon\varphi\rangle_{J_n}}\rangle_{J_n}\to F(\varepsilon)$. Since the left-hand side is bounded from above by $\varepsilon^p\|\varphi\|^p_{\mathrm{BMO}^p(Q)}$, we have

$$\varepsilon^p\|\varphi\|^p_{\mathrm{BMO}^p(Q)}\ge b_{p,F(\varepsilon)}\big(0,F(\varepsilon)\big).$$

Now, take $\varepsilon\to\varepsilon_\varphi$ (and, thus, $F(\varepsilon)\to\infty$). This gives

$$\varepsilon_\varphi^p\|\varphi\|^p_{\mathrm{BMO}^p(Q)}\ge\varepsilon_0^p(p).$$

Take any $f\in L^\infty(Q)$, then $\varepsilon_{\varphi-f}=\varepsilon_\varphi$. Thus, we can replace φ with $\varphi-f$ above, which gives

$$\|\varphi-f\|_{\mathrm{BMO}^p(Q)}\ge\frac{\varepsilon_0(p)}{\varepsilon_\varphi}.$$

The same inequality holds with φ replaced with $-\varphi$, and it remains to take the infimum over $f\in L^\infty(Q)$ on the left. □

5. Optimizers and the converse inequality

In this section, we complete the proof of Theorem 3.1 by proving Lemma 3.3. To that end, we present a set of special test functions on the interval $(0,1)$ that realize the infimum in definition (1.10) of the Bellman function $\boldsymbol{b}_{p,C}$.

Without loss of generality assume $C>1$. Let $Q=(0,1)$. Recall the Bellman candidate $b_{p,C}$ given by formula (2.18). For $x\in\Omega_C$, we say that a function φ_x on Q is an optimizer for $b_{p,C}$ at x if

$$\varphi_x\in E_{x,C,Q}\quad\text{and}\quad\langle|\varphi_x|^p\rangle_Q=b_{p,C}(x),\tag{5.1}$$

where the set of test functions $E_{x,C,Q}$ is defined by (1.9). Observe that if we have such a function φ_x for all $x\in\Omega_C$, then

$$b_{p,C}(x)=\langle|\varphi|^p\rangle_Q\ge\boldsymbol{b}_{p,C}(x),$$

which is the statement of Lemma 3.3.

Our optimizers φ_x will have different forms depending on the location of x in Ω_C. Specifically, we will have a different optimizer for each of the four subdomains R_k of Ω_C defined by formula (1.8) and pictured in Figure 2. We do not discuss the construction of these optimizers, but simply give formulas for them. A reader interested in where they come from is invited to consult papers [10] and [3], where a number of similar constructions are carried out in the context of BMO^2.

For each $x\in R_1$, let

$$\varphi_x(t)=u+\xi\log\left(\tfrac{\alpha}{t}\right)\chi_{(0,\alpha)}(t), \tag{5.2}$$

where $u=u(x)$ is defined by (2.2) and we set

$$\alpha=\frac{x_1-u}{\xi}. \tag{5.3}$$

(This optimizer was defined in Section 5 of [8] under the name φ_x^+.)

Now consider the subdomain R_2. Let us give names to its four corners, clockwise from top right:

$$X=(\overline{w}+\xi,e^{\overline{w}+\xi}),\qquad Y=(\overline{w},e^{\overline{w}}),\qquad Z=(\bar v,e^{\bar v}),\qquad W=(\bar v+\xi,e^{\bar v+\xi}).$$

We already know the optimizers for the points X, Y, and Z : the first comes from formula (5.2) (which applies since $X\in R_1\cap R_2$) with $\alpha=1$; the other two are trivial, since for each $x\in\Gamma_1$, the set $E_{x,C,Q}$ contains only one element — the constant function $\varphi(t)=x_1$. Therefore, we define, for all $t\in Q$,

$$\varphi_X(t)=\xi\log\left(\tfrac{1}{t}\right),\qquad \varphi_Y(t)=\overline{w},\qquad \varphi_Z(t)=\bar v.$$

We now use these three optimizers to define φ_x for every $x\in R_2$. Observe that R_2 is contained in the triangle with the vertices X, Y, Z. This means that every x in R_2 has a unique representation as a convex combination of these three points. Thus, there exist non-negative numbers α_1, α_2, and α_3 such that $\alpha_1+\alpha_2+\alpha_3=1$ and

$$x=\alpha_1X+\alpha_2Y+\alpha_3Z. \tag{5.4}$$

To obtain φ_x, we concatenate φ_X, φ_Y, and φ_Z in the appropriate proportion:

$$\varphi_x(t)=\varphi_X\left(\tfrac{t}{\alpha_1}\right)\chi_{(0,\alpha_1)}(t)+\varphi_Y\left(\tfrac{t-\alpha_1}{\alpha_2}\right)\chi_{(\alpha_1,\alpha_1+\alpha_2)}(t)+\varphi_Z\left(\tfrac{t-\alpha_1-\alpha_2}{\alpha_3}\right)\chi_{(\alpha_1+\alpha_2,1)}(t),$$

or, equivalently,

$$\varphi_x(t)=\overline{w}\,\chi_{(0,\alpha_1+\alpha_2)}(t)+\xi\log\left(\tfrac{\alpha_1}{t}\right)\chi_{(0,\alpha_1)}(t)+\bar v\,\chi_{(\alpha_1+\alpha_2,1)}(t), \tag{5.5}$$

with $\alpha_k=\alpha_k(x)$ defined by (5.4).

This formula applies, in particular, to the fourth corner of R_2, i.e., the point W. That point also lies in the subdomain R_3, and is the key to defining the optimizer for all other points in that subdomain. Specifically, with the knowledge of φ_W we define the optimizer φ_x for an arbitrary point $x\in R_3$ by the formula

$$\varphi_x(t)=\varphi_W\left(\tfrac{t}{\tau\alpha}\right)\chi_{(0,\tau\alpha)}(t)+\xi\log\left(\tfrac{\alpha}{t}\right)\chi_{(\tau\alpha,\alpha)}(t)+u\chi_{(\tau\alpha,1)}(t). \tag{5.6}$$

Here, u is given by (2.2), α is given by (5.3), and we also set

$$\tau=e^{(u-\bar v)/\xi}. \tag{5.7}$$

It remains to define φ_x when $x\in R_4$. Recall the two auxiliary functions $v=v(x)$ and $w=w(x)$ defined by Lemma 2.6 (see Figure 3). Every point $x\in R_4\backslash\Gamma_1$ lies on the line segment connecting the points (v,e^v) and (w,e^w). Accordingly, we define φ_x to be the appropriate concatenation of the two constant optimizers corresponding to those points:

$$\varphi_x(t)=w\,\chi_{(0,\beta)}(t)+v\,\chi_{(\beta,1)}(t), \tag{5.8}$$

where we set

$$\beta=\frac{x_1-v}{w-v}. \tag{5.9}$$

The following lemma immediately yields Lemma 3.3.

Lemma 5.1. *Let* φ_x *be defined by* (5.2) *and* (5.3) *for* $x \in R_1$; *by* (5.4) *and* (5.5) *for* $x \in R_2$; *by* (5.6), (5.3), *and* (5.7) *for* $x \in R_3$; *and by* (5.8) *and* (5.9) *for* $x \in R_4$. *Then* φ_x *is an optimizer for* $\mathbf{b}_{p,C}$ *at every* $x \in \Omega_C$.

Remark 5.2. If a point x lies on a boundary shared by two subdomains, φ_x seems to be defined by two different formulas. However, as is easy to check, in all cases above, such two formulas give exactly the same function.

The proof of this lemma is very similar to the proof of Lemma 5.2 from [8] and we leave it to the reader.

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University of Cincinnati
E-mail address: `leonid.slavin@uc.edu`

St. Petersburg Department of the V. A. Steklov Mathematical Institute, RAS, and St. Petersburg State University
E-mail address: `vasyunin@pdmi.ras.ru`